

复变函数 2023年B卷参考答案

(15') 1. $f(z) = 3x^2y - y^3 + A + i(-x^3 + 3xy^2 + A)$, 这里 A 为任意实常数
或 $f(z) = -iz^2 + (1+i)A$, $A \in \mathbb{R}$.

(15') 2. $f(z) = \frac{\operatorname{Ln} z}{z^2 - 1}$ 的各解析分支为 $f_k(z) = \frac{\ln z + 2k\pi i}{z^2 - 1}$, ($k = 0, \pm 1, \dots$).

$z=1$ 为 $f_0(z)$ 的可去奇点, 为 $f_k(z)$ 的一阶极点 ($k = \pm 1, \dots$).

$$\operatorname{Res}_{z=1} f_0(z) = 0 \quad \operatorname{Res}_{z=1} f_k(z) = k\pi i. \quad (k = \pm 1, \pm 2, \dots).$$

(10') 3.
$$\int_{|z|=\frac{5}{2}} \frac{1}{(z-3)(z^5-1)} dz$$

$$= -2\pi i \left[\operatorname{Res}_{z=3} \frac{1}{(z-3)(z^5-1)} + \operatorname{Res}_{z=\infty} \frac{1}{(z-3)(z^5-1)} \right]$$

$$= -2\pi i \left[\frac{1}{3^5-1} + \operatorname{Res}_{t=0} \frac{1}{(\frac{1}{t}-3)(\frac{1}{t^5}-1)} \frac{1}{t^2} \right]$$

$$= -2\pi i \left(\frac{1}{242} + 0 \right)$$

$$= -\frac{\pi i}{121}.$$

(10') 4.
$$I = \int_0^{2\pi} \frac{t \sin \theta}{1 - 2t \cos \theta + t^2} \nu(\cos \theta, \sin \theta) d\theta$$

$$= \int_0^{2\pi} \frac{t \sin \theta}{1 - 2t \cos \theta + t^2} \operatorname{Im} f(e^{i\theta}) d\theta$$

$$= \operatorname{Im} \left(\int_0^{2\pi} \frac{t \sin \theta}{1 - 2t \cos \theta + t^2} f(e^{i\theta}) d\theta \right)$$

$$z = e^{i\theta} \\ \equiv \operatorname{Im} \left(\int_{|z|=1} \frac{t \frac{z-z^{-1}}{2i}}{1 - 2t \frac{z+z^{-1}}{2} + t^2} \cdot f(z) \cdot \frac{1}{iz} dz \right)$$

$$= \operatorname{Im} \left(\int_{|z|=1} \frac{t}{2} \cdot \frac{z^2-1}{z} \cdot \frac{f(z)}{(tz-1)(z-t)} dz \right)$$

由 $f(z)$ 在 $|z| \leq 1$ 解析, $-1 < t < 1$, 知

$$I = \operatorname{Im} \left(2\pi i \left(\operatorname{Res}_{z=0} \frac{t(z^2-1)f(z)}{2z(tz-1)(z-t)} + \operatorname{Res}_{z=t} \frac{t(z^2-1)f(z)}{2z(tz-1)(z-t)} \right) \right)$$

$$= \operatorname{Im} \left(2\pi i \left(-\frac{f(0)}{2} + \frac{f(t)}{2} \right) \right)$$

$$= \operatorname{Im} (\pi i \cdot f(t))$$

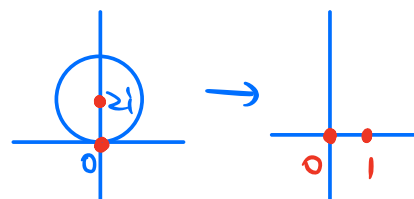
$$= \pi f(t).$$

(10') 5. 易知 $w=f(z)$ 可为分式线性变换,

由保对称点性知 $w=-1$ 时 $z=\infty$.

故 $(0, 1, -1, w) = (0, 2i, \infty, z)$,

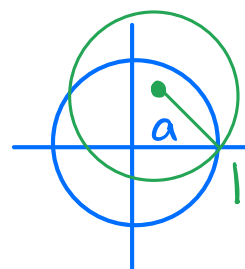
$$\text{即 } \frac{w}{w-1} : \frac{-1}{-1-1} = \frac{z}{z-2i} : \frac{1}{1}, \text{ 得 } w = -\frac{z}{z-4i}.$$



$$(10') 6. f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{1}{1-a-(z-a)} = \frac{1}{1-a} \cdot \frac{1}{1-\frac{z-a}{1-a}} \\ = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(1-a)^{n+1}}, \quad |z-a| < |1-a|.$$

当 $|a| < 1$, $a \notin [0, 1)$ 时,

$f(z)$ 在 a 处的 Taylor 级数 $\sum_{n=0}^{\infty} \frac{(z-a)^n}{(1-a)^{n+1}}$ 是 $f(z)$ 的解析延拓.



(10') 7. $f(z)$ 在 $z=a$ 处的 Taylor 展式:

$$f(z) = f(a) + f'(a)(z-a) + \dots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \dots$$

$$\text{记 } D = \{z \in \mathbb{C} \mid |z| < 1\},$$

$$E_0 = \{a \in D \mid f(a) = 0\}, \dots, E_n = \{a \in D \mid f^{(n)}(a) = 0\}, \dots$$

由于 $f(z)$ 为超越整函数,

故 $f^{(n)}(z)$ 在 $\overline{D}$ 上解析且不恒为常数.

从而由零点的孤立性, E_n 为有限点集 ($n=0,1,2,\dots$)

$$\text{又 } \bigcup_{n=0}^{\infty} E_n \neq D,$$

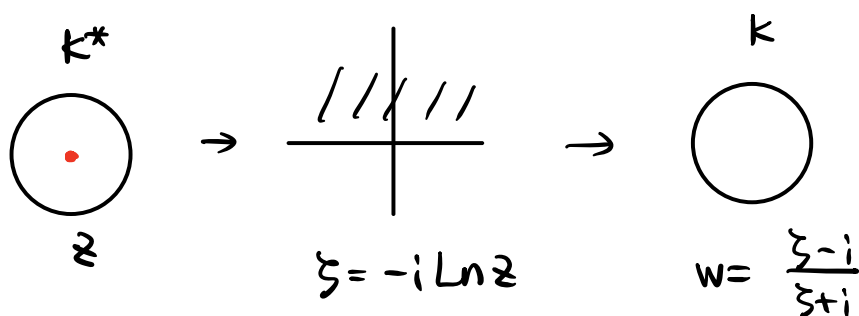
故 $\exists b \in D, b \notin E_n (\forall n=0,1,2,\dots)$ 即 $f^{(n)}(b) \neq 0 (\forall n=0,1,2,\dots)$

于是 $f(z)$ 在 $z=b$ 处的所有 Taylor 系数非零.

(10') 8. (1) 不存在.

因为单叶解析函数将单连通区域映成单连通区域.

(2) 存在. 例子:



$$w = f(z) = \frac{-i \operatorname{Ln} z - i}{-i \operatorname{Ln} z + i} = \frac{\operatorname{Ln} z + 1}{\operatorname{Ln} z - 1}.$$

(101) 9. ① 记 $f(z)$ 在 H 中的零点为 a_k ($k=1, \dots, 2023$).

由 $L_k(z) = \frac{z - a_k}{z - \bar{a}_k}$ 是从 H 到 $\{w \mid |w| < 1\}$ 上的共形映射,

知 $|L_k(z)| < 1$, $z \in H$.

$$\text{令 } g(z) = \frac{f(z)}{\prod_k L_k(z)} = f(z) \cdot \prod_k \frac{z - \bar{a}_k}{z - a_k},$$

则 g 在 H 中解析, 且 a_k 不是 g 的零点.

$$\Rightarrow \begin{cases} |g(a_k)| > 0 = |f(a_k)| \\ |g(z)| = \frac{|f(z)|}{\prod_k |L_k(z)|} > |f(z)|, \quad z \neq a_k, z \in H \end{cases}$$

$$\Rightarrow |g(z)| > |f(z)|, \quad \forall z \in H.$$

$$\textcircled{2} \text{ 下证 } \sup_{z \in H} |g(z)| = \sup_{z \in H} |f(z)|.$$

由 L_k 在 $\bar{H}$ 连续, 且当 $z \in \partial H$ 时, $|L_k(z)| = 1$, 可得

$$\forall \varepsilon > 0, \exists \delta > 0, \forall z \in H_\delta = \{z \in H \mid \operatorname{Im} z < \delta \text{ 或 } |z| > \frac{1}{\delta}\},$$

$$|L_k(z)| > 1 - \varepsilon, \quad k=1, \dots, 2023.$$

$$\Rightarrow |f(z)| = |g(z)| \prod_k |L_k(z)| > (1 - \varepsilon)^{2023} |g(z)|, \quad z \in H_\delta.$$

$$\text{由最大模原理, } \sup_{z \in H} |g(z)| = \sup_{z \in H_\delta} |g(z)|, \quad \forall \delta > 0.$$

$$\text{故 } \sup_{z \in H} |f(z)| \geq (1 - \varepsilon)^{2023} \sup_{z \in H_\delta} |g(z)| = (1 - \varepsilon)^{2023} \sup_{z \in H} |g(z)|, \quad \forall \varepsilon > 0.$$

$$\Rightarrow \sup_{z \in H} |f(z)| \geq \sup_{z \in H} |g(z)|.$$

$$\text{又由 } \textcircled{1} \text{ 知 } \sup_{z \in H} |g(z)| \geq \sup_{z \in H} |f(z)|, \text{ 从而得 } \sup_{z \in H} |g(z)| = \sup_{z \in H} |f(z)|.$$