

四川大学期末考试试题 B 卷参考答案
(复变函数 2024——2025 学年第 2 学期)

第 1 题解答: $w = \operatorname{Ln} z = \ln|z| + i(\arg z + 2k\pi), k \in \mathbb{Z}, -\pi < \arg z < \pi.$

由 $w(1) = 2\pi i$ 知 $k = 1$. 因此 $w(i) = \ln 1 + i\left(\frac{\pi}{2} + 2\pi\right) = \frac{5}{2}\pi i.$

第 2 题解答: (1) $f = e^x(x \cos y + ay \sin y) + ie^x(bx \sin y + cy \cos y).$

$$u_x = e^x(x \cos y + ay \sin y + \cos y), \quad v_x = e^x(bx \sin y + cy \cos y + b \sin y),$$

$$u_y = e^x(-x \sin y + ay \cos y + a \sin y), \quad v_y = e^x(bx \cos y - cy \sin y + c \cos y).$$

由 C.-R. 方程: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ 在复平面上成立可得 $a = -1, b = c = 1.$

第 3 题解答: 方法一: $f(z) = \frac{2z^2 + 13z + 20}{(z+3)^2(z+2)}$ 在 $|z| < 2$ 内解析. 由于 $c_1 = f'(0)$, 并且

$$f'(z) = \frac{4z+13}{(z+3)^2(z+2)} - (2z^2+13z+20) \frac{2(z+3)(z+2) + (z+3)^2}{(z+3)^4(z+2)^2}.$$

$$\text{所以 } c_1 = f'(0) = \frac{13}{18} - 20 \times \frac{12+9}{3^4 \times 4} = -\frac{13}{54}.$$

$$\text{方法二: } f(z) = \frac{2z^2 + 13z + 20}{(z+3)^2(z+2)} = \frac{(2z+5)(z+4)}{(z+3)^2(z+2)} = \frac{1}{(z+3)^2} + \frac{2}{z+2}.$$

$$\begin{aligned} \text{当 } |z| < 2 \text{ 时, } \frac{2}{z+2} &= \frac{1}{1+\frac{z}{2}} = \sum_{n=0}^{\infty} \left(-\frac{z}{2}\right)^n, \quad \frac{1}{(z+3)^2} = -\left(\frac{1}{z+3}\right)' = -\left(\frac{1}{3} \times \frac{1}{1+\frac{z}{3}}\right)' \\ &= -\left(\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n\right)' = -\frac{1}{3} \sum_{n=1}^{\infty} \frac{n z^{n-1}}{3^n} = -\sum_{n=0}^{\infty} \frac{(n+1) z^n}{3^{n+2}}. \end{aligned}$$

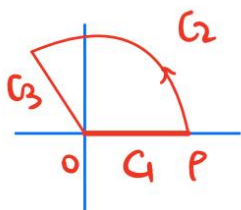
$$\text{因此, } f(z) = \sum_{n=0}^{\infty} \left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty} \frac{(n+1) z^n}{3^{n+2}}. \quad n=1 \Rightarrow c_1 = -\frac{1}{2} - \frac{2}{3^2} = -\frac{31}{54}.$$

第 4 题解答:

$$(1) \oint_{|z|=2} \frac{5z+1}{(z-1)(z+1)^2} dz = 2\pi i \left(\left. \frac{5z+1}{(z+1)^2} \right|_{z=1} + \left(\left. \frac{5z+1}{z-1} \right)' \right|_{z=-1} \right) = 2\pi i \left(\frac{6}{4} - \frac{5}{2} + \frac{4}{4} \right) = 0.$$

$$\begin{aligned} \text{或者 } \oint_{|z|=2} \frac{5z+1}{(z-1)(z+1)^2} dz &= 2\pi i \left(\operatorname{Res}_{z=1} \frac{5z+1}{(z-1)(z+1)^2} + \operatorname{Res}_{z=-1} \frac{5z+1}{(z-1)(z+1)^2} \right) \\ &= -2\pi i \operatorname{Res}_{z=\infty} \frac{5z+1}{(z-1)(z+1)^2} = -2\pi i \operatorname{Res}_{t=0} \frac{5\frac{1}{t}+1}{\left(\frac{1}{t}-1\right)\left(\frac{1}{t}+1\right)^2} \times \frac{1}{t^2} = -2\pi i \operatorname{Res}_{t=0} \frac{5+t}{(1-t)(1+t)^2} = 0. \end{aligned}$$

第 5 题解答:



$\frac{z^3}{z^6+1}$ 的奇点有 $z_k = e^{\frac{2k\pi+\pi}{6}i}$, $k=0,1,2,3,4,5$. 它们均为一阶极点.

$\frac{z^3}{z^6+1}$ 的奇点在周线 C (ρ 充分大) 内的有 $e^{\frac{\pi}{6}i}, e^{\frac{5\pi}{6}i}$.

$$\text{由于 } \operatorname{Res}_{z=z_k} \frac{z^3}{z^6+1} = \frac{z^3}{(z^6+1)'} \Big|_{z=z_k} = \frac{z^3}{6z^5} \Big|_{z=z_k} = \frac{1}{6z^2} \Big|_{z=z_k}.$$

$$\text{故 } \oint_C \frac{z^3}{z^6+1} dz = 2\pi i \left(\operatorname{Res}_{z=e^{\frac{\pi}{6}i}} \frac{z^3}{z^6+1} + \operatorname{Res}_{z=e^{\frac{5\pi}{6}i}} \frac{z^3}{z^6+1} \right) = 2\pi i \left(\frac{1}{6e^{\frac{\pi}{3}i}} - \frac{1}{6} \right).$$

$$\text{在 } C_3 \text{ 上, 记 } z = re^{\frac{2\pi}{3}i}, \text{ 则 } \int_{C_3} \frac{z^3}{z^6+1} dz = -\int_0^\rho \frac{r^3 e^{2\pi i}}{r^6 e^{4\pi i} + 1} e^{\frac{2\pi}{3}i} dr = -e^{\frac{2\pi}{3}i} \int_0^\rho \frac{r^3}{r^6+1} dr.$$

$$\text{故由 } \oint_C \frac{z^3}{z^6+1} dz = \int_{C_1} \frac{z^3}{z^6+1} dz + \int_{C_2} \frac{z^3}{z^6+1} dz + \int_{C_3} \frac{z^3}{z^6+1} dz, \text{ 令 } \rho \rightarrow +\infty,$$

$$\text{得 } \frac{2\pi i}{6} \left(e^{-\frac{\pi}{3}i} - 1 \right) = \int_0^{+\infty} \frac{r^3}{r^6+1} dr + 0 - e^{\frac{2\pi}{3}i} \int_0^{+\infty} \frac{r^3}{r^6+1} dr = \left(1 - e^{\frac{2\pi}{3}i} \right) \int_0^{+\infty} \frac{r^3}{r^6+1} dr.$$

$$\text{从而 } \int_0^{+\infty} \frac{r^3}{r^6+1} dr = \frac{2\pi i}{6} \frac{e^{-\frac{\pi}{3}i} - 1}{1 - e^{\frac{2\pi}{3}i}} = \frac{\pi}{3} i \cdot \frac{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}{\frac{3}{2} - \frac{\sqrt{3}}{2}i} = \frac{\pi}{3\sqrt{3}}.$$

第 6 题解答:

假设 ∞ 是 $f(z)$ 的本质奇点, 则存在复平面中点列 $\{z_n\}$ 满足 $z_n \rightarrow \infty$ ($n \rightarrow \infty$), 并且 $f(z_n) \rightarrow f(0)$ ($n \rightarrow \infty$).

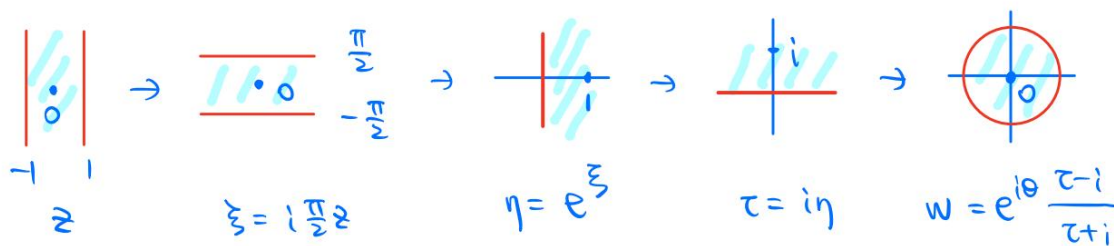
依据反函数定理, $f^{-1}(z)$ 也是解析函数, 从而

$$z_n = f^{-1}(f(z_n)) \rightarrow f^{-1}(f(0)) = 0 \quad (n \rightarrow \infty).$$

上式与 $z_n \rightarrow \infty$ 矛盾. 因此, ∞ 是 $f(z)$ 的可去奇点或者极点, 即 $f(z)$ 是一个多项式.

又因为 $f(z)$ 具有单叶性, 所以 $f(z)$ 只能是一个一次多项式.

第7题解答:



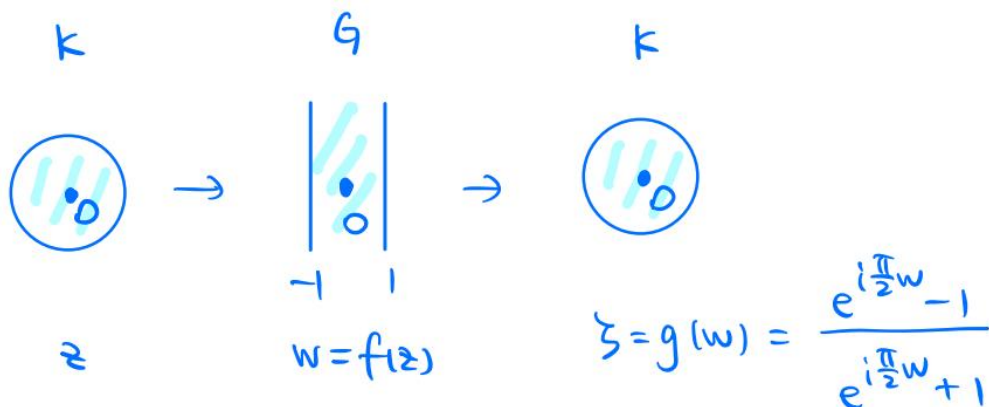
如图, $w = f(z) = w \circ \tau \circ \eta \circ \xi(z)$ 将 G 共形映射到 K , 并且 $f(0) = 0$.

又因为 $f'(z) = w'(\tau)\tau'(\eta)\eta'(\xi)\xi'(z) = e^{i\theta} \left(\frac{1}{\tau+i} - \frac{\tau-i}{(\tau+i)^2} \right) \cdot i \cdot e^\xi \cdot i \frac{\pi}{2}$.

$f'(0) = w'(i)\tau'(1)\eta'(0)\xi'(0) = e^{i\theta} \frac{1}{2i} \cdot i \cdot e^0 \cdot i \frac{\pi}{2} = \frac{\pi}{4} e^{i(\theta+\frac{\pi}{2})} > 0$. 所以 $\theta = -\frac{\pi}{2}$.

因此 $w = f(z) = e^{-\frac{\pi}{2}i} \frac{i\eta-i}{i\eta+i} = \frac{\eta-1}{\eta+1} = \frac{e^{\frac{i\pi}{2}z}-1}{e^{\frac{i\pi}{2}z}+1}$ 就是所求的共形映射.

第8题解答:



利用第7题的结论, 取 $g(w) = \frac{e^{\frac{i\pi}{2}w}-1}{e^{\frac{i\pi}{2}w}+1}$ 将 G 共形地映射成 K ,

并且 $g(0) = 0, g'(0) = \frac{\pi}{4}$.

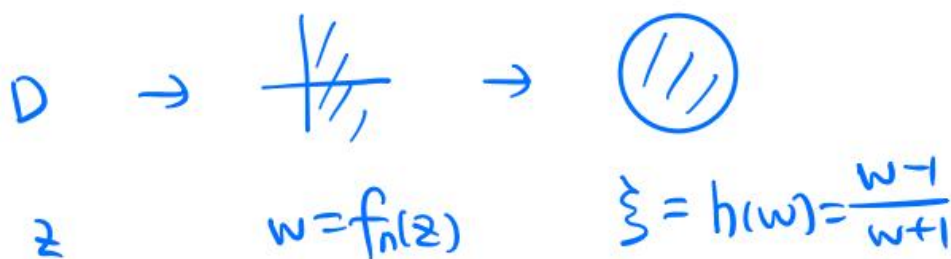
由于 $w = f(z)$ 将 K 映射到 G 内, $f(0) = 0$, 因此 $\zeta(z) = g(f(z))$ 将 K 映射到 K 内,

并且 $\zeta(0) = g(f(0)) = 0$.

由 *Schwarz* 引理, $|\zeta'(0)| \leq 1$.

又因为 $\zeta'(0) = g'(0)f'(0) = \frac{\pi}{4}f'(0)$, 所以 $|f'(0)| \leq \frac{4}{\pi}$.

第9题解答:



(1) 取 $\xi = h(w) = \frac{w-1}{w+1}$, 则 $w = h^{-1}(\xi) = \frac{1+\xi}{1-\xi}$ 的奇点为 $\xi = 1$.

(2) 记 $g_n = h \circ f_n = \frac{f_n - 1}{f_n + 1}$, 则 $|g_n| < 1$, $f_n = h^{-1} \circ g_n = \frac{1+g_n}{1-g_n}$.

(i) 若 g_n 内闭一致收敛于 1, 即 $\forall$ 闭集 $\bar{F} \subset D$,

$$\forall \varepsilon > 0, \exists N > 0, s.t. \forall n > N, \forall z \in \bar{F}, |g_n - 1| < \varepsilon,$$

则 $\forall M > 0$, 取 $\varepsilon = \frac{2}{M+1}$, $\exists N > 0, s.t. \forall n > N, \forall z \in \bar{F}$,

$$|f_n| = \left| \frac{1+g_n}{1-g_n} \right| = \left| \frac{2}{1-g_n} - 1 \right| \geq \frac{2}{|1-g_n|} - 1 > \frac{2}{\varepsilon} - 1 = M,$$

即 f_n 内闭一致发散于 ∞ , 矛盾. 故 g_n 不内闭一致收敛于 1.

(ii) 由 (i), $\exists \bar{F} \subset D, \exists \varepsilon_0 > 0, s.t. \forall k, \exists n_k > N, \exists z_k \in \bar{F}, |g_{n_k}(z_k) - 1| \geq \varepsilon_0$.

对于该子列 g_{n_k} , 由 *Montel* 定理, 在 D 内有内闭一致收敛的子列.

不妨仍记为 g_{n_k} , 记 $g_{n_k} \rightarrow g$. 由 *Weierstrass* 定理, g 在 D 内解析.

由于 $z_k \in \bar{F}$, 故 z_k 存在聚点 $z_0 \in \bar{F}$, 于是 $|g(z_0) - 1| \geq \varepsilon_0 > 0$, $g(z_0) \neq 1$.

(iii) 证明 $\forall a \in D, g(a) \neq 1$.

由 (ii) 知, $g(z)$ 不恒等于 1, 因此利用零点的孤立性, 可知 $\exists \rho > 0, N_\rho(a)$ 的闭包包含于 D 内, s.t. 在 $C: |z-a|=\rho$ 上, $g(z)-1|_C \neq 0$. 记 $m = \min_{z \in C} |g(z)-1| > 0$.

由于 $g_{n_k}(z)$ 在 C 上一致收敛于 $g(z)$, 故 $\exists K > 0, \text{s.t. } \forall k > K, |g_{n_k}(z) - g(z)| < m \leq |g(z)-1|, \forall z \in C$. 由 *Rouche* 定理, $N(g(z)-1, C) = N(g_{n_k}(z)-1, C)$.

然而 $|g_{n_k}(z)| < 1, g_{n_k}(z)-1$ 在 C 内无零点, 故 $g(z)-1$ 在 C 内无零点. 因此, $g(a) \neq 1$.

(3) 由 (iii) 知, $g(z)-1|_D \neq 0$. 于是对于 $\forall \bar{F} \subset D, \|g(z)-1\|_{\bar{F}} \geq \delta = \min_{z \in \bar{F}} |g(z)-1| > 0$.

由于 g_{n_k} 内闭一致收敛于 $g: \forall \varepsilon > 0 (\varepsilon < \frac{\delta}{2}), \exists K > 0, \forall k > K, |g_{n_k} - g| < \varepsilon$.

于是 $|g_{n_k} - 1| = |g_{n_k} - g + g - 1| \geq |g-1| - |g_{n_k} - g| \geq \delta - \varepsilon > \frac{\delta}{2}$.

$|f_{n_k}| = \left| \frac{1+g_{n_k}}{1-g_{n_k}} \right| \leq \frac{1+|g_{n_k}|}{\frac{\delta}{2}} \leq \frac{4}{\delta}$ 一致有界, $\forall z \in \bar{F}$.

故由 *Montel* 定理, $\{f_{n_k}\}$ 有内闭一致收敛的子列.